Coordination sequences and information spreading in small-world networks

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We study the spread of information in small-world networks generated from different d-dimensional regular lattices, with $d=1$, 2, and 3. With this purpose, we analyze by numerical simulations the behavior of the coordination sequence, e.g., the average number of sites $C(n)$ that can be reached from a given node of the network in n steps along its bonds. For sufficiently large networks, we find an asymptotic behavior $C(n) \sim \rho^n$, with a constant ρ that depends on the network dimension d and on the rewiring probability p (which measures the disorder strength of a given network). A simple model of information spreading in these networks is studied, assuming that only a fraction q of the network sites are active. The number of active nodes reached in n steps has an asymptotic form λ^n, λ being a constant that depends on p and q, as well as on the dimension d of the underlying lattice. The information spreading presents two different regimes depending on the value of λ: For $\lambda>1$ the information propagates along the whole system, and for $\lambda<1$ the spreading is damped and the information remains confined in a limited region of the network. We discuss the connection of these results with site percolation in small-world networks.

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I. INTRODUCTION

It is well known that the structure of communication and transportation systems plays a crucial role in the spread of information. Also, the structure of social networks is important for the dynamics of disease propagation. In last years, several models of these complex networks have been introduced motivated by empirical data in different fields [1–3]. In particular, networks showing the ''small-world'' effect have attracted a great deal of attention. In these networks, the ''distance'' between any two elements is small as compared with the system size, and the propagation of information (or signal, disease, damage, . . .) takes place much faster than in regular lattices [4–8].

Small-world networks are well suited to study properties of physical systems with underlying networks ranging from regular lattices to random graphs [9,10], by changing a single parameter [11]. Watts and Strogatz [4] have proposed a model for social networks, which is based on a locally highly connected regular lattice, in which a fraction p of the links between nearest-neighbor sites are randomly replaced by new random links, thus creating long-range ''shortcuts.''

In these networks, one has at the same time a local neighborhood (as for regular lattices) and some global properties of random graphs. The small-world effect is usually measured by the scaling behavior of the characteristic path length ℓ, defined as the average of the distance between any two sites. For a random network one has a logarithmic increase of ℓ with the network size N (number of sites), while for a d-dimensional regular lattice one finds an algebraic dependence: $\ell \sim N^{1/d}$. In small-world networks, there is a crossover size $N^* \sim p^{-1}$ that separates the large- and small-world regimes [12–14], and the small-world behavior appears for any finite value of p $(0<p<1)$ as soon as the network is large enough.

This shorter global length scale changes strongly the behavior of statistical physical problems on small-world networks, as compared with regular lattices. Among these problems, one finds in the literature signal propagation [4], spread of infections [15,16], and random spreading of information [17,18]. Site and bond percolation [16,19,20], as well as the ferromagnetic Ising model [21,22], have been also studied in these networks.

Up to now, most of the published work on small worlds has concentrated on networks obtained from one-dimensional lattices (rings). Small-world networks built by rewiring lattices of higher dimensions have being employed by Newman and Watts [19] to study percolation, as a model of disease propagation. Also, damage spreading has been studied for Ising models on small-world networks obtained from two-dimensional (2D) and three-dimensional (3D) lattices [23].

In this paper we investigate the spread of information in small-world networks built up from different d-dimensional regular lattices (with $d=1$, 2, and 3), which allows us to analyze the influence of the dimensionality on the long-distance characteristics of these networks. With this goal, we study by numerical simulations the asymptotic behavior of the so-called ''coordination sequence,'' a concept usually employed to characterize the connectivity of crystal lattices [24–26]. It gives information about the number of sites $C(n)$ that can be reached from a given site in a number of steps n (in one step one goes from a site to a nearest neighbor). A similar quantity has been studied earlier in small-world networks by means of analytical calculations in a continuum model [19,27,28], as it is relevant for the analysis of spreading processes in these networks. From our numerical simulations, we find for large networks the asymptotic behavior $C(n) \sim \rho^n$, with a constant $\rho>1$ that depends on the rewiring probability p and on the network dimension d. Then, we study a simple model of information spreading, assuming that only a fraction of the network sites are active sites (which receive and propagate the information). The number of active sites reached in n steps has an asymptotic form

$\sim\lambda^n$, being λ a constant that depends on p and on the fraction of active sites. We discuss the connection of our results with site percolation in these networks.

The paper is organized as follows. In Sec. II we describe the procedure employed to generate the networks studied here, as well as some of their characteristics. In Sec. III we present results for the coordination sequences, and in Sec. IV we analyze the spread of information in small-world networks. The paper closes with some conclusions in Sec. V.

II. SMALL-WORLD MODEL

The networks studied here were generated from regular lattices of different dimensions: 1D ring with coordination number four, 2D square, and 3D cubic lattices. Small-world networks were built up according to the model of Watts and Strogatz [4,11]. We consider in turn each of the bonds in the starting lattice and replace it with a given probability p by a new bond. This means that one end of the bond is changed to a new node chosen at random from the whole network. One imposes the conditions that no two nodes can have more than one bond connecting them, and no node can be connected by a link to itself. This method keeps constant the total number of links in the rewired networks. Thus the average coordination number z in the 1D and 2D cases amounts to 4, and in our 3D networks $z=6$. The total number of rewired links is $\frac{1}{2}zpN$ on average. In the rewiring process we avoided isolated sites (with zero links), and thus each site has at least one neighbor. With this procedure we obtained networks in which more than 99.9% of the sites were connected in a single component. (A random graph has usually different components of various sizes.) Note that there is a technical difference between our procedure and that of Watts and Strogatz [4], in which each site has at least $z/2$ neighbors in the rewired network.

The size of the networks used in our calculations was larger than the crossover size N^* [14,19], so that we were in the small-world regime. Our largest networks included 3×10^5 sites for $d=1$, 500×500 sites for $d=2$, and $80\times80\times80$ for $d=3$. Periodic boundary conditions were assumed.

For regular lattices all nodes have the same connectivity, i.e. the same number of nearest neighbors. However, for $p>0$ different connectivities are possible, giving rise to a probability distribution that we will call $P(m)$. For a (large) random network with average coordination number z, $P(m)$ is given by a Poisson distribution [2,9]

$$P_{rd}(m)=\frac{z^m e^{-z}}{m!}. \tag{1}$$

For the small-world networks studied here, one can find an analytic expression for the probability distribution of connectivities, $P_{an}(m)$. To do this, we will distinguish between the ends of links that remain on their original sites (as in the regular lattice), and those changed in the rewiring process. The number of links in a network with mean coordination z amounts to $\frac{1}{2}zN$. Since for each link connecting two sites in the starting regular lattice, we change one of these sites with probability p, the average number of changes per site is $\frac{1}{2}zp$. This means that each connection of a site is removed in the rewiring process with probability $t=p/2$. Hence, the probability distribution of the original (not rewired) links starting from a site is given by

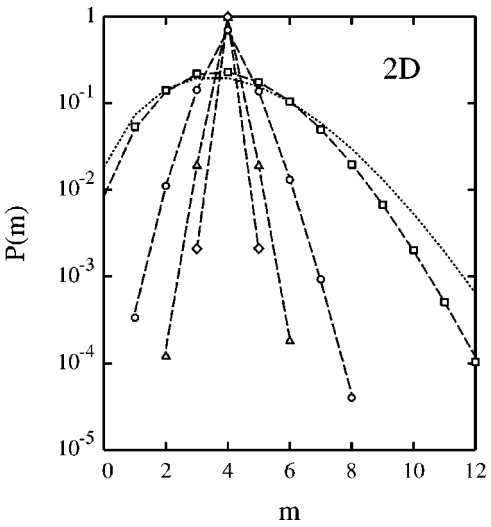

FIG. 1. Probability distribution of the connectivity m for small-world networks. Symbols are the results of numerical simulations for networks generated from a 2D lattice of size 500×500: diamonds, $p=0.001$; triangles, $p=0.01$; circles, $p=0.1$; squares, $p=1$. Dashed lines are guides to the eye, obtained from $P_{an}(m)$ values derived by using Eq. (4). A dotted line shows the probability distribution $P_{rd}(m)$ for a random network with mean connectivity $z=4$ [Eq. (1)].

$$P_1(r)=\binom{z}{r}(1-t)^r t^{z-r}, \tag{2}$$

for $r=0,\ldots,z$. On the other side, the distribution of the number of links generated in the rewiring process is, in the limit of large N

$$P_2(s)=\frac{1}{s!}(tz)^s e^{-zt}, \tag{3}$$

for $s\geqslant0$. Finally, the probability distribution for the connectivities in small-world networks is given by

$$P_{an}(m)=\sum_{r=0}^{r_{max}}\binom{z}{r}(1-t)^r t^{z-r}(tz)^{m-r}\frac{e^{-tz}}{(m-r)!}, \tag{4}$$

for $m\geqslant0$ and $r_{max}=\min(z,m)$. This expression is similar to that derived by Barrat and Weigt [21] for the networks generated by these authors. (The main difference is that they left untouched $z/2$ links per site, as in Ref. [4].) The probability distribution $P_{an}(m)$ for $z=4$ is represented in Fig. 1 by dashed lines joining points corresponding to different m. Each line corresponds to a particular value of the rewiring probability p. Symbols indicate results of numerical simulations for small-world networks generated from a 2D square lattice: diamonds, $p=0.001$; triangles, $p=0.01$; circles, $p=0.1$; squares, $p=1$. Note that the simulated networks do

not include isolated sites, i.e., $P(0)=0$. This fact, however, does not affect a good agreement between simulated and analytical results, since $P_{an}(0)$ derived from Eq. (4) is small, and even for $p=1$ we have $P_{an}(0)<0.01$.

As expected, the distribution $P(m)$ becomes broader as p increases. For $p=1$, our results (squares) approach the connectivity distribution of a random network with $z=4$ (dotted line), which is given by Eq. (1). Nevertheless, it turns out that both probability distributions are clearly different. This is due to the fact that our networks with $p=1$ are not random networks, as they keep memory of the starting regular lattices. The main reason for this memory effect is that we rewire only one end of each link, maintaining the other end on its original site.

III. COORDINATION SEQUENCES

Coordination sequences $C(n)$ count the number of nodes that can be reached from a given point of the network by a number n of steps along its bonds. $C(n)$ is thus the number of nodes in the ''coordination shell'' n. This concept generalizes the familiar coordination number (also called connectivity, or vertex degree in graph theory), which is the first member of this series $(n=1)$. For regular lattices, the coordination sequence $C(n)$ is a polynomial. For the lattices considered here one has $C_{1D}(n)=4$, $C_{2D}(n)=4n$, and $C_{3D}(n)=4n^2+2$. In general, for d-dimensional lattices, $C(n)$ is a polynomial of degree $d-1$. On the other side, for a (large) random network with mean coordination z, the average coordination sequence is given by $C_{rd}(n)=z^n$ (see the Appendix). Note that for random and small-world networks the coordination sequence depends on the considered starting node of the network. In the sequel we will call $C(n)$ the average coordination sequence, i.e., the mean value obtained (for each n) by averaging the coordination sequences of all network sites.

In Fig. 2 we give the average coordination sequences $C(n)$ for 2D small-world networks of different sizes $L\times L$, with L from 50 to 500, and for rewiring probability $p=0.01$. $C(n)$ increases with the distance n up to a maximum value C_{max}, which depends on the system size, and then decreases for larger n. For each L one observes a region in which $\log C(n)\sim n$, before reaching saturation at C_{max}. Such a region becomes larger as L rises, and leads to an asymptotic dependence of the form: $C(n)\sim\rho^n$ with a constant $\rho>1$ (dashed line in Fig. 2). This exponential dependence on n contrasts with the power law corresponding to the starting regular lattice.

A similar behavior is found for $C(n)$ in 1D and 3D networks, and for different values of p. This is shown in Fig. 3 for small-world networks built up from lattices of different dimensions: (a) $d=1$, (b) $d=2$, and (c) $d=3$. In all cases one finds for sufficiently large n the dependence $C(n)\sim\rho^n$, which is similar to the expression corresponding to a random network with mean connectivity ρ (instead of the actual mean connectivity z). Hence, we will call ρ ''effective connectivity'' of the considered network, which will depend on the dimension d and on the rewiring probability p. This effective connectivity controls the long-range properties of the

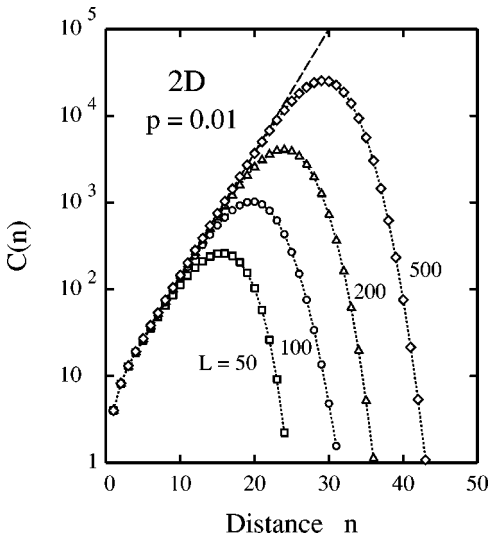

FIG. 2. Average coordination sequence $C(n)$ for 2D networks of different sizes $L\times L$ and $p=0.01$. Different symbols represent several values of L: squares, $L=50$; circles, $L=100$; triangles, $L=200$; diamonds, $L=500$. The dashed line is an extrapolation for large L. Dotted lines are guides to the eye.

networks under consideration. By increasing n, one observes in the 2D and 3D cases a crossover from the regular-lattice behavior (power-law dependence, small n) to an exponential dependence (large n). This crossover takes place at a distance n_0, which increases for decreasing p, and is found to scale as $n_0\sim p^{-1/d}$, as expected from earlier calculations [19]. In 1D we do not observe such a crossover, since for the regular lattice one has $C(n)=4$ (a constant), and the exponential dependence appears already for small n values, even for $p=0.001$ [squares in Fig. 3(a)].

The mean distance between pairs of sites ℓ in a given network (the characteristic path length) is given in our context by the average value $\langle n\rangle$, calculated with the normalized probability distribution $C(n)/N$. As indicated above, this average distance ℓ is known to increase logarithmically with the system size. This dependence of ℓ appears naturally from the asymptotic dependence of $C(n)$, as for $C(n)\sim\rho^n$ one has

$$\ell\sim\frac{\log N}{\log\rho}. \tag{5}$$

Here one can see again the role of ρ as an effective connectivity for small-world networks, since for random networks with mean connectivity z one has $\ell\sim\log N/\log z$ [2]. For $p=1$ we obtain ρ values close to (but lower than) the mean connectivity z, as expected from the fact that in this case we have networks that are not totally random (for which one should have $\rho=z$).

For a given dimension d, the effective connectivity ρ decreases as p is lowered. In the limit $p\to 0$ one has $\rho\to 1$, as expected when one approaches a regular lattice. In Fig. 4 we present the effective connectivity ρ as a function of p in a double logarithmic plot. One observes that $\rho-1$ follows for small p a power law of the form: $\rho-1=a\,p^b$, with constants

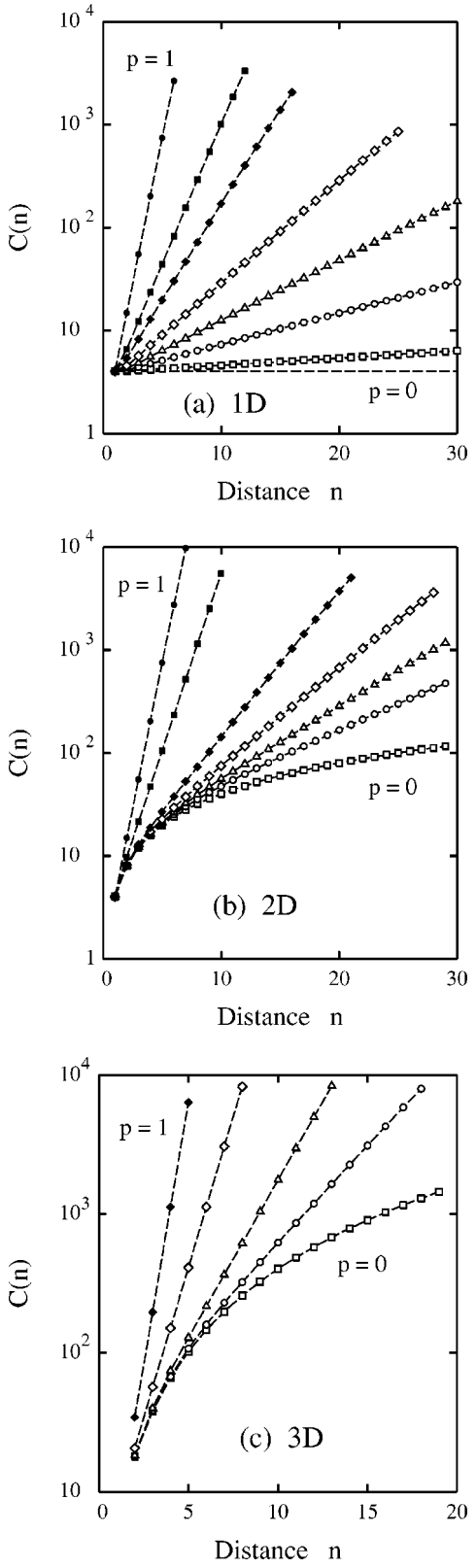

FIG. 3. Average coordination sequence $C(n)$ for networks of different dimension d, and for several values of the rewiring probability p. (a) 1D networks with 10^5 sites; from top to bottom, $p=1$, 0.1, 0.05, 0.02, 0.01, 0.005, 0.001, and 0. (b) 2D networks of size 500×500, with $p=1$, 0.1, 0.01, 0.004, 0.002, 0.001, and 0. (c) 3D networks of size $80\times80\times80$, with $p=1$, 0.05, 0.005, 0.001, and 0. Dashed lines are guides to the eye.

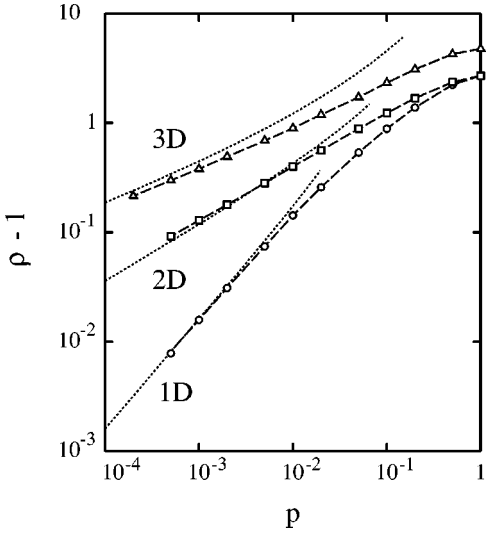

FIG. 4. Effective connectivity ρ for small-world networks generated from different d-dimensional lattices. We present $\rho-1$ vs the rewiring probability p in a double-logarithmic plot for 1D (squares), 2D (circles), and 3D (triangles) networks. Dashed lines are guides to the eye. Dotted lines correspond to ρ_{th} values obtained from analytical calculations in a continuum model [19,27,28].

a and b dependent on the network dimension. By numerical fitting, we find for the exponent b the values 0.98(2), 0.49(2), and 0.36(2), in 1D, 2D, and 3D, respectively.

To understand this dependence on p of the effective connectivity, we will consider the length scale characteristic of small-world networks, given by the typical distance between ends of shortcuts [19]:

$$\xi=(pz)^{-1/d}. \tag{6}$$

First, we note that for regular lattices, one has $C(n+1)/C(n)\to1$ for large n. Second, in the presence of a low concentration of shortcuts one expects [19,27]

$$C(n+1)\approx C(n)[1+B/\xi], \tag{7}$$

B being a constant dependent on the network topology. Therefore, for small p we find for ρ a dependence in the form

$$\rho=1+ap^{1/d}, \tag{8}$$

in good agreement with the results of our numerical simulations. In particular, we find for the exponent b defined above: $b=1/d$. Our numerical result for 3D networks is slightly higher than $1/d=0.33$, which seems to indicate that still smaller p values (larger network sizes) are necessary to improve the agreement with this prediction.

In Fig. 4 we compare these numerical results for the effective connectivity ρ with those obtained from analytic calculations in a continuum model (dotted lines) [19,27,28]. These calculations allowed to obtain analytic expressions for the "volume" $V(r)$, which gives the average number of neighbors of a given node in a neighborhood of "radius" r. In particular, our $C(n)$ should coincide with $A(r)$ (the "surface area"), defined as the derivative $A(r)=dV(r)/dr$ in

Refs. [19,28], when r takes integer values. Note, however, that the small-world networks employed in those works were generated in a way different from those used here. These authors, instead of rewiring each bond with probability p, added shortcuts between pairs of sites chosen uniformly at random, without removing any bonds from the starting lattice. This procedure results to be more convenient for analytical calculations, but does not keep constant the mean coordination, which in this case increases with p. For 1D small-world networks with $z=4$, such analytical procedure gives in the limit $N\to\infty$,

$$A(r)=4\exp(16pr), \tag{9}$$

from where one has in our notation $\rho_{th}=\exp(16p)$. This result is displayed by a dotted line in Fig. 4, and agrees with our numerical results for small p values (represented by circles). For $p>0.01$ the analytical result is higher than that derived from our simulations, since the mean connectivity corresponding to a particular p value is larger in the former approach.

For $d=2$ and 3, the continuum model [27] gives in the limit of large networks,

$$\rho_{th}=\exp\{[2p\Gamma_d(d-1)!]^{1/d}\}, \tag{10}$$

with $\Gamma_2=2\pi$ and $\Gamma_3=4\pi$. This analytical dependence of ρ_{th} is close to the results of our simulations for small p in 2D and 3D networks. The agreement is, however, not so good as for $d=1$. For $d=2$ it seems that both results cross at $p\approx 5\times 10^{-3}$, suggesting that the continuum approach gives p values slightly lower than the actual discrete networks for $p\to 0$. It is not clear from our results whether something similar happens for $d=3$, in which case p values smaller than those employed in our simulations would be necessary. In any case, taking into account the differences between both methods, the continuum model gives a good description of the asymptotic behavior of the coordination sequence $C(n)$ for large n.

IV. SPREAD OF INFORMATION

We now turn to the question of information spreading in small-world networks. We will study a simple model, that can be equally useful to analyze propagation of signal, disease, or damage in these networks. Given a certain information starting from a single site, our discussion will be basically concentrated to studying the number of sites that receive such information in a number of steps n. We assume that a fraction q of the sites are inactive, i.e. that they cannot receive the information and therefore do not propagate it. These sites are taken randomly in the network. We consider the simplest case that in a time unit the information propagates from an active site to all its active nearest neighbors. In the language of epidemiology, our active sites are ''susceptible'' individuals, and the ''transmissibility'' of our model is unity (every contact between a healthy but susceptible individual and an infected individual results in transmission). This model is similar to that employed in Refs. [16,19] to study site percolation in these networks. For $q=0$ (all sites are active) the average number of active receivers in step n, $S(n)$, coincides with the above studied $C(n)$. For a given network and a particular distance n, this average number $S(n)$ will decrease as the fraction q of inactive sites rises. For large enough q, one expects that the information spreading at long distances will be blocked. This is expected to happen when the fraction of active sites $1-q$ is lower than the percolation threshold of the considered network.

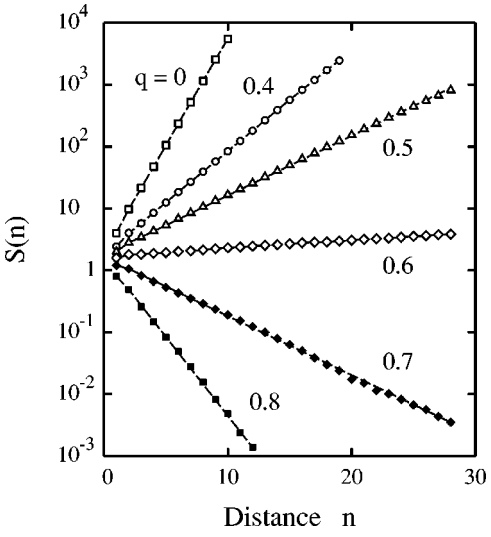

FIG. 5. Average number $S(n)$ of active nodes reached in n steps, as derived from numerical simulations for 2D small-world networks of size 500×500 and rewiring probability $p=0.1$. Different symbols correspond to several values of the fraction of inactive nodes q, as indicated by labels. Dashed lines are guides to the eye.

In Fig. 5 we show the average number $S(n)$ of active receivers as a function of the distance n, for a particular 2D network corresponding to $p=0.1$. In this plot, different symbols indicate various q values. For each value of q, $S(n)$ depends on n exponentially, in the form

$$S(n)=c\lambda^n, \tag{11}$$

with constants c and λ. We will call λ ''propagation constant,'' since it controls the long-range character of the information spreading. The value of λ decreases from the effective connectivity ρ of the network (for $q=0$), to zero in the limit $q\to 1$. This exponential dependence of $S(n)$ appears for sufficiently large n in all considered cases ($d=1,2,3$, as well as different p values), and in general the above constants c and λ depend on p, q, and the network dimension d. For 2D and 3D networks, the value n_0 necessary to find an exponential law increases for decreasing p, similarly to the case of the coordination sequences $C(n)$ discussed above [see Figs. 3(b) and 3(c)].

For $\lambda>1$, the information starting from a single active site propagates through the network, and eventually reaches the whole system without decaying, irrespective of the system size N. On the contrary, for $\lambda<1$ the spread of information is damped and it cannot propagate to arbitrary distances in the network. The propagation constant λ is presented in Fig. 6, as derived from numerical simulations for (a) 1D and

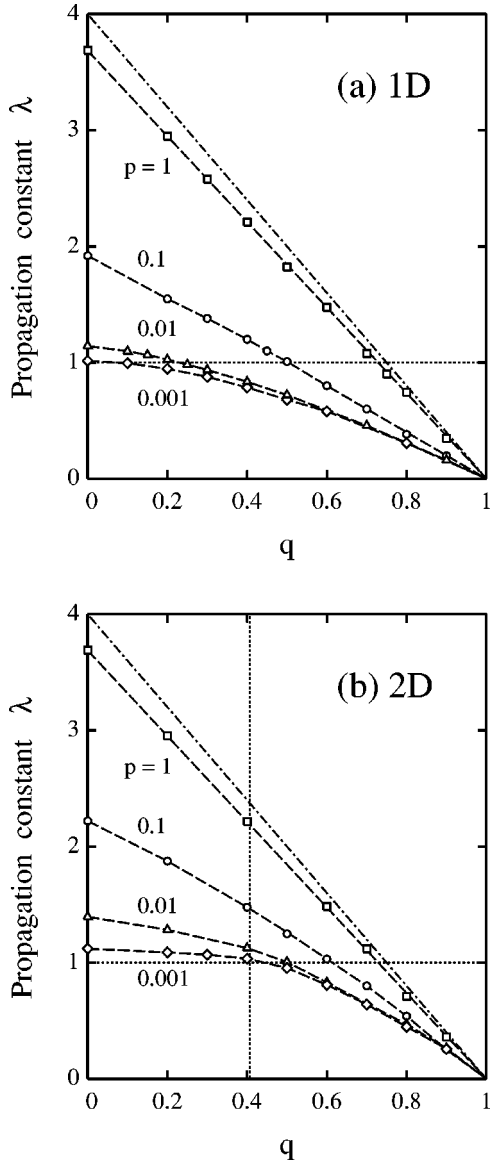

FIG. 6. Propagation constant λ as a function of the fraction of inactive nodes q, for different values of the rewiring probability p. (a) 1D networks; (b) 2D networks. Dashed lines are guides to the eye. Dash-dotted lines correspond to random networks with mean connectivity $z=4$. The vertical dotted line in (b) indicates the fraction q_c of inactive nodes, corresponding to the percolation threshold for active sites in a 2D square lattice $(1-q_c)$.

(b) 2D networks. In each case we give results for several values of the rewiring probability p. For a given p, λ decreases as the fraction of inactive sites q rises, and there is a crossing point $q_c(p)$ from propagation of information to damping, for which $\lambda=1$. For $d=3$ we found results qualitatively similar to those obtained for $d=2$.

For a random network one trivially has: $\lambda_{rd}=z(1-q)$ [dash-dotted lines in Figs. 6(a) and 6(b)], since the active sites and the connections between them form a new random network with mean coordination $z'=z(1-q)$. Therefore, one has in this case $\lambda=1$ for $q_c=1-1/z$. This critical value agrees with that derived in Ref. [29] for resilience of random networks (with a Poisson connectivity distribution) to random breakdown of nodes. In that case, q_c is the critical fraction of nodes that needs to be removed from a random network before it disintegrates.

For our small-world networks with $p=1$, the dependence of λ on q can be approximated by

$$\lambda_{p=1}\approx\rho(1-q), \tag{12}$$

ρ being the effective connectivity. This means that close to $p=1$, the crossing point appears at $q_c\approx 1-1/\rho$ for our small-world networks. However, for smaller p the function $\lambda(q)$ is no longer linear on q. This can be clearly seen in the limit $p\to 0$, where the function $\lambda(q)$ flattens close to $q=0$ (for small fraction of inactive sites). This is the behavior expected for regular lattices, where one should have $\lambda=1$ in a region $0<q<q_c$. In 2D and 3D networks with $p=0$, the difference $1-q_c$ coincides with the percolation threshold for active sites in the corresponding regular lattices. In 1D regular networks, this region collapses onto a single point, because in this case $q_c=0$. In general, for any value of the rewiring probability p, a propagation constant $\lambda>1$ requires the presence of clusters of active sites with an extent on the order of the typical length of the system. This means that the fraction of active sites has to be larger than the percolation threshold of the considered network. Then, for given p and dimension d, the crossing point q_c appears for a fraction of active nodes equal to the site percolation threshold. In the language of disease propagation, q_c is the minimum concentration of immune individuals necessary to avoid an epidemic. In the language of network robustness, it is the point at which a large enough number of nodes have been deleted (or damaged) from a communication network to prevent communication on large scales [10].

For $\lambda<1$ and large enough systems, the total number of active receivers is given by

$$N_{rec}=\sum_{n=1}^{n_{max}} S(n)\sim\frac{c\lambda}{1-\lambda}, \tag{13}$$

which obviously diverges for $\lambda\to 1$. Close to the crossing point one can expand λ to first order in $q-q_c$,

$$\lambda=1+\alpha(q-q_c), \tag{14}$$

with

$$\alpha=\left.\frac{d\lambda}{dq}\right|_{q_c}<0, \tag{15}$$

which is valid for any $p>0$. Then, we find for $q>q_c$,

$$N_{rec}\sim\frac{c}{|\alpha|(q-q_c)}. \tag{16}$$

Thus, for $q\to q_c^+$ the number of active receivers N_{rec} diverges as $(q-q_c)^{-1}$. This agrees with the scaling law for the mean size of percolating clusters in small-world networks. In fact, close to the site percolation threshold one finds a critical

exponent $\sigma=1$ [20] (this critical exponent appears also for site percolation in regular lattices when $d\to\infty$ [30]).

A quantitative measure of the extent of the network region that receives the information can be obtained by calculating the mean distance l_{pr} from the starting node to all the active receivers. This propagation length l_{pr} is given by the average value $\langle n\rangle$ calculated with the probability distribution $S(n)/N_{rec}$, and one finds for $\lambda<1$, $l_{pr}\sim 1/(1-\lambda)$. For a given network of size N, the information will reach the whole system when $l_{pr}\approx\ell$ (the characteristic path length of the network). On the contrary, when $l_{pr}<\ell$ the information is confined in a limited region of the network.

We finally note that values for the site percolation threshold that can be derived from our crossing point at $\lambda=1$ do not coincide with those found in earlier works [16,19,20]. As discussed above in connection with coordination sequences, the reason for this apparent disagreement is that the small-world networks employed in those works were generated in a way different from those used here. Therefore, site percolation thresholds obtained in Refs. [16,19,20] are smaller than those derived with our procedure (for $p>0$), as in the former case the mean connectivity z' is larger than that of the starting lattice: $z'=z(1+p)$.

V. CONCLUSIONS

Coordination sequences are a suitable tool to characterize the long-range behavior of small-world networks. For large networks and distances $n>\xi$ (the characteristic length of the network), the sequence $C(n)$ increases as ρ^n, thus showing a functional dependence similar to random networks. The effective connectivity ρ ranges from unity in the limit of regular lattices to the mean connectivity z of the network for random graphs. For small p this effective connectivity follows a dependence $\rho=1+ap^{1/d}$, being d the dimension of the underlying lattice.

A simple model of information spreading in these networks has been studied, assuming that a fraction q of the network nodes are inactive. For large n, the number of active receivers in n steps scales as λ^n, with a propagation constant $\lambda\leqslant\rho$ that depends on the disorder (rewiring probability) and q. The information spreading shows two different regimes depending on the value of λ. On one side, for $\lambda>1$ the information starting from a single site propagates along a cluster of interconnected active sites, that spans the whole system. On the other side, for $\lambda<1$ the spreading is damped, and the information remains confined in a limited region of the network. The crossover from one regime to the other appears at a point where the concentration of active sites $1-q_c$ coincides with the percolation threshold of the considered network.

Our results support the conjecture of Ref. [20] that the critical exponents for percolation in small-world networks do not depend on the dimension of the underlying regular lattice. In fact, from the long-range connectivity studied here one expects that all critical properties of these networks will be in the same universality class as random graphs. This is also in line with results for Ising [21,31] and XY models [32] in small-world networks, which display order-disorder transitions of mean-field type, as happens for random networks.

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APPENDIX: COORDINATION SEQUENCES FOR RANDOM NETWORKS

In this appendix we derive the mean coordination sequence $C_{rd}(n)$ for random networks. For a random network with mean connectivity z, the connectivity distribution $P_{rd}(m)$ is given by a Poisson distribution, as in Eq. (1). For $n=1$ we have obviously $C_{rd}(1)=z$.

Given a generic node and a link starting on it, we call $Q_{rd}(m)$ the connectivity distribution for the other end of the link. The probability of reaching a node with connectivity m is proportional to m. Hence

$$Q_{rd}(m)=\frac{1}{z}mP_{rd}(m), \tag{A1}$$

where z in the denominator is a normalization factor. Then, the average number of next-nearest neighbors is given by

$$C_{rd}(2)=z\sum_{m>1}(m-1)Q_{rd}(m). \tag{A2}$$

By inserting the Poisson distribution for $P_{rd}(m)$ in Eqs. (A1) and (A2) one finds $C_{rd}(2)=z^2$. Using the same procedure for $n>2$,

$$C_{rd}(n)=C_{rd}(n-1)\sum_{m>1}(m-1)Q_{rd}(m)=zC_{rd}(n-1), \tag{A3}$$

and consequently

$$C_{rd}(n)=z^n. \tag{A4}$$

Note that this is valid in the limit of large networks, as we assume that nodes in different coordination shells are different, and each node in shell n is connected to the starting node by a single n-steps path. This means that for a given n, the probability of finding loops with $n'\leqslant n$ members is negligible for our purposes.

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